Tunneling in quantum wires II: A new line of IR fixed points.

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In a previous paper, we showed that the problem of tunneling in quantum wires was integrable in the isotropic case $g_\sigma = 2$. In the present work, we continue the exploration of the general phase diagram by looking for other integrable cases. Specifically, we discuss in details the manifold $g_\rho + g_\sigma = 2$, where the associated "double sine-Gordon" model is integrable. Transport properties are exactly computed. Surprisingly, the IR fixed points, while having complete reflection of charge and spin currents, do not correspond to two separate leads. Their main characteristic is that they are approached along irrelevant operators of dimension $1 + \frac{1}{g_\rho}$ and $1 + \frac{1}{g_\sigma}$, corresponding to transfer of one electron charge but no spin, or one spin 1/2 but no charge.

I. INTRODUCTION.

While much progress has been made recently in the field of quantum impurity problems, it is still an open challenge to fully understand the phase diagram for a single (charge) impurity in a one dimensional quantum wire, where the electrons are described by a Luttinger liquid with charge and spin degrees of freedom.

The problem was studied using the renormalisation group and perturbation theory by Kane and Fisher [1] and Furusaki and Nagasoa [2]. A complete picture of the flows generated by the RG was obtained in terms of the "g" factors, g_ρ and g_σ, describing the interaction of bulk charge and spin degrees of freedom. The existence of tentalizing new fixed points, which would be partly transmitting and partly reflecting, was in particular conjectured. While steps were taken to identify these fixed points using conformal field theory [3], the determination of their exact nature - in particular the computation of the universal associated conductances - remains open.

In this paper, we are addressing this impurity problem from the point of view of integrability. A simpler problem, the case of an impurity in a spinless electron gas, was mapped on the boundary sine-gordon model [4], which is a completely integrable quantum field theory [5]. In our case, the problem of electrons with spin and impurity, it maps after bosonization and the same folding maneuvers as [4] on a "double sine-Gordon model", with two coupling constants β_1 and β_2 (related simply to g_ρ and g_σ). Unfortunately, this model does not seem to be generically integrable. There are however special integrable manifolds. In the first part of this work [6] the case $g_\sigma = 2$ was solved by using a slightly different theory than the double sine-Gordon model, with however the same boundary interaction. Here a more direct route is adopted and we investigate the integrability of the "double sine-Gordon models". The simplest one is given by $g_\rho + g_\sigma = 2$, and can be considered as a particular case of a new integrable model recently discovered by Fateev [7].

This integrable manifold is, a priori, not the most interesting, since it lies far away from the region where the new IR fixed points are conjectured to arise. However, a closer study along the lines of [4] gives rise to very intriguing results. By a variety of arguments, including the computation of the non equilibrium tunneling current at vanishing temperature, we found that the IR fixed point does **not** correspond to two separate leads ("open" fixed point [3]), as suggested by a (maybe too naive) analysis of the classical action [1]. It is not completely clear to us what this IR fixed point is. Although it corresponds to complete reflection (vanishing transmitted current) for spin and charge, the operators along which it is approached seem to correspond to collective excitations that would be impossible for separate leads: the leading operators describe actually either transfer of one electron charge but no spin, or one spin 1/2 but no charge!

The paper is organized as follows. Section 2 is a brief summary of the field theoretic description of the problem and of the folding of [4]. Section 3 studies the "double sine-Gordon model" in full generality. The integrable manifold is studied using the technique of non local conserved currents of [8], and S and R matrices are obtained. The IR fixed point is (partially) identified and shown to differ from the expected one. In section 4, the out of equilibrium current with an external potential is computed exactly. Section 5 contains our conclusions. Many additional technical details are given in appendices. In appendix A, we explain why the model, although quantum integrable, is presumably not classically integrable. This is in sharp contrast with the case of the ordinary sine-Gordon model, and suggests that semi classical arguments as in [1] are maybe not reliable. Appendix B discusses the problem of quantum boundary integrability. In appendix C, a perturbative computation of the S matrix around the free fermion point is presented, which confirms our non perturbative identification based on non local currents. Appendix D is a first order Keldysh computation that fixes the relation between the bare coupling in our action and the parameter T_B of the R matrix.

Appendix E sketches the computation of the whole two point current correlator using form-factors, which allows a more complete identification of the operator content of the IR fixed point.

II. THE MODEL.

In order to fix notations and the problem, let us repeat some of the definitions in [1]. In the one dimensional Luttinger liquid, bosonisation of the fermionic operators is accomplished via

$$\psi_\mu^\dagger \simeq \sum_{n \ odd} e^{in[\sqrt{\pi}\Theta_\mu + k_F x]} e^{i\sqrt{\pi}\phi_\mu(x)}, \tag{1}$$

with $\mu = \uparrow, \downarrow$. The fields ϕ_μ and Θ_μ have commutation relations

$$[\phi_\mu(x), \Theta_{\mu'}(x')] = i\delta_{\mu,\mu'}\theta(x - x') \tag{2}$$

from which we can devise two different representations of the Luttinger liquid (our conventions follow [1]). Here, we will work in the so called Θ-representation. Changing basis to the charge and spin degrees of freedom

$$\Theta_\rho = \Theta_\uparrow + \Theta_\downarrow, \quad \Theta_\sigma = \Theta_\uparrow - \Theta_\downarrow, \tag{3}$$

we obtain the action

$$S = S_\rho + S_\sigma \tag{4}$$
$$= \int dxdy \frac{1}{2g_\rho}[(\partial_x\Theta_\rho)^2 + (\partial_y\Theta_\rho)^2] + \frac{1}{2g_\sigma}[(\partial_x\Theta_\sigma)^2 + (\partial_y\Theta_\sigma)^2].$$

where y is the imaginary time, the spin and charge velocities have been normalised such that $v_\sigma = v_\rho = 1$. In this convention, the g-factors have value $g_\sigma = g_\rho = 2$ for a non-interacting system.

The electric and magnetic conductances of this system follow directly from Kubo's formula

$$G_{\rho/\sigma} = \frac{e^2}{h}g_{\rho/\sigma}. \tag{5}$$

As discussed in [9], this formula is actually not correct for the charge conductance, because of the way the system is coupled to the reservoirs. One rather has

$$G_\rho = 2\frac{e^2}{h}, \tag{6}$$

for any g_ρ. In this paper, we compute the conductance as in [1] to make comparison with the results of this paper more straightforward. The correct physical results follow by simple rescaling.

We are interested in the physical properties of this model in the presence of a charge impurity at the origin $x = 0$. Due to this impurity, the hamiltonian gets an additional piece

$$\delta H = \int dx V(x)(\psi_\uparrow^\dagger \psi_\uparrow + \psi_\downarrow^\dagger \psi_\downarrow). \tag{7}$$

Here $V(x)$ is a potential which has essentially zero measure outside $x = 0$ and under the bosonisation rules sketched at the beginning, this leads to the change of action

$$\delta S = \lambda \int dy \cos \sqrt{\pi}\Theta_\rho \cos \sqrt{\pi}\Theta_\sigma, \tag{8}$$

where $\lambda \simeq V(2k_F)$. In general, all terms consistent with the symmetries of the action must be considered at the boundary since they will be generated by renormalization. This means that the most general form of the perturbation is

$$\delta S = \int dy \sum_{\substack{n_\rho, n_\sigma \\ n_\rho + n_\sigma \ even}} \frac{\lambda_{n_\rho, n_\sigma}}{4} e^{i\sqrt{\pi}(n_\rho\Theta_\rho + n_\sigma\Theta_\sigma)} \tag{9}$$

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where the λ's are real couplings. The renormalisation group equations read at first order

$$\frac{d\lambda_{n_\rho,n_\sigma}}{dl} = \left(1 - \frac{n_\rho^2}{4}g_\rho - \frac{n_\sigma^2}{4}g_\sigma\right)\lambda_{n_\rho,n_\sigma}. \tag{10}$$

In this paper, for reasons given later, we will restrict to the manifold

$$g_\sigma + g_\rho = 2, \tag{11}$$

so that only the perturbation with $n_\rho = n_\sigma = 1$ is relevant. In the following we denote λ_{11} by λ.

In [4], the equivalent problem for spinless electrons was exactly solved. The solution required a folding to transform the impurity problem into a boundary problem, and then used recent results on boundary integrable quantum field theories [5] together with the massless scattering approach. The same folding can easily be acomplished in the problem with spin. First, it is convenient to rescale the fields, writing the action as

$$S = S_\rho + S_\sigma + \lambda \int dy \cos \sqrt{\pi g_\rho}\Theta_\rho \cos \sqrt{\pi g_\sigma}\Theta_\sigma \tag{12}$$

$$S_\mu = \int dxdy \frac{1}{2}[(\partial_x\Theta_\mu)^2 + (\partial_y\Theta_\mu)^2]. \tag{13}$$

We then introduce odd and even fields

$$\Theta_\mu^e = \frac{1}{\sqrt{2}}[\Theta_{\mu,L}(x,y) + \Theta_{\mu,R}(-x,y)] \tag{14}$$

$$\Theta_\mu^o = \frac{1}{\sqrt{2}}[\Theta_{\mu,L}(x,y) - \Theta_{\mu,R}(-x,y)]. $$

With this, the interaction at $x = 0$ involves only the even fields which have Neumann boundary conditions. The odd fields, having Dirichlet boundary conditions, completely decouple and do not interact. In the previous form, the even field is left moving and we can "fold" using

$$\theta_{\mu,L} = \Theta_\mu^e(x,y), \ x < 0$$
$$\theta_{\mu,R} = \Theta_\mu^e(-x,y), \ x < 0, \tag{15}$$

and now expressing everything in terms of the fields θ which is defined on the negative axis, we get the action

$$S = S_0 + \mathcal{B}$$
$$= \int_{-\infty}^{0} dx \int dy \frac{1}{2} \sum_{\mu=\sigma,\rho} [(\partial_x\theta_\mu)^2 + (\partial_y\theta_\mu)^2] + \lambda \int dy \cos\sqrt{\frac{\pi g_\rho}{2}}\theta_\rho(0) \cos\sqrt{\frac{\pi g_\sigma}{2}}\theta_\sigma(0). \tag{16}$$

The next step in [4] was to use the integrability of the corresponding boundary quantum field theory - the boundary sine-Gordon model (BSG). In the present case with spin, the problem, involving two fields, is more complicated, and very few results are available. We must first investigate in more details this "double sine-Gordon model" (DSG) and its boundary counterpart (BDSG). The following section, together with a series of appendices, are devoted to that study. The reader interested in physical properties only can go directly to the end of next section.

III. THE DOUBLE SINE-GORDON MODEL

A. Non local currents and integrable manifold

Although the problem at hand involves an interaction only at the boundary, it is crucial - like for the ordinary sine-Gordon model [10] - to understand first the model with a bulk interaction. We start thus with the general action for the double sine-Gordon model, with two fields ϕ_1, ϕ_2

$$S = S_0 + S_{bulk}$$
$$= S_0 + \Lambda \int dxdy \cos\beta_1\phi_1 \cos\beta_2\phi_2. \tag{17}$$

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A more general form of this action was first investigated in [7], where an integrable manifold was identified, of which our model is a subset. The following arguments, in particular the use of affine quantum group symmetries, are ours, but were strongly influenced by the results of [7]. The double cosine term (the perturbation) has dimension $x = \frac{\beta_1^2 + \beta_2^2}{4\pi}$. The free propagators are $\langle \phi_i(z, \bar{z}) \phi_j(z', \bar{z}') \rangle = -\frac{1}{4\pi} \delta_{ij} \ln |z - z'|^2$, where we set $z = y - ix$.

Unfortunately, we do not expect the model to be integrable for any choice of the coupling constants β_1, β_2. A useful strategy to identify quantum integrable varieties is to look for non-local conserved currents, following Bernard and Leclair [8]. A current $J(z)$ will be conserved if it is local with respect to the perturbation, and if the residue of the simple pole in the short distance expansion of J and the perturbation is a total derivative. By trial and error, the simplest possible candidates have the form

$$J_i^{\pm}(z) = \partial \varphi_i \, e^{\pm i \frac{4\pi}{\beta_j} \varphi_j}, \tag{18}$$

where $\varphi_i(\bar{\varphi})$ are the chiral and antichiral components of the field, and in (18), $i \neq j$. One has for instance (exponentials are implicitly normal ordered)

$$\partial \varphi_1(z) e^{-i \frac{4\pi}{\beta_2} \varphi_2(z)} e^{i[\beta_1 \varphi_1(w) + \beta_2 \varphi_2(w)]} = \frac{1}{z - w} e^{i[\beta_1 \varphi_1(w) + \beta_2 \varphi_2(w) - \frac{4\pi}{\beta_2} \varphi_2(z)]} \left[-\frac{i\beta_1}{4\pi(z - w)} + \partial \varphi_1 \right]$$

so the simple pole has residue

$$\left(-\frac{\beta_1}{\beta_2} \partial \varphi_2 + \partial \varphi_1 \right) e^{i[\beta_1 \varphi_1 + (\beta_2 - \frac{4\pi}{\beta_2}) \varphi_2]}.$$

This is a total derivative iff $-\frac{\beta_1^2}{\beta_2} = \beta_2 - \frac{4\pi}{\beta_2}$ or

$$\beta_1^2 + \beta_2^2 = 4\pi. \tag{19}$$

In that case, the perturbation has physical dimension $d = 1$ [1]. Setting

$$\bar{\partial} J_i^{\pm} = \partial H_i^{\pm}; \ \partial \bar{J}_i^{\pm} = \bar{\partial} \bar{H}_i^{\pm}, \tag{20}$$

the currents give rise to conserved charges

$$Q_i^{\pm} \propto \int dz J_i^{\pm} + \int d\bar{z} H_i^{\pm}$$

$$\bar{Q}_i^{\pm} \propto \int d\bar{z} \bar{J}_i^{\pm} + \int dz \bar{H}_i^{\pm}. \tag{21}$$

By counting arguments, following [11], one expects that the conservation extends to all orders in perturbation theory [2]. In addition, we of course have two conserved topological charges [3]

$$T_i = \frac{\beta_i}{\pi} \int_{-\infty}^{\infty} \partial_x \phi. \tag{22}$$

The invariance of the action constrains T_1 and T_2 to have the same parity.

Following the construction of [8], one checks that the non local conserved charges generate the quantum affine algebras $\widehat{sl}_{q_i}(2)$, with

$$q_i = -\exp\left(-i \frac{2\pi^2}{\beta_i^2} \right). \tag{23}$$

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[1] In the case of the ordinary sine-Gordon model, this dimension corresponds to the free fermion point. The manifold (19) does not, however, correspond to free fermions except when $\beta_i^2 = 2\pi$.

[2] The fact that the dimension of the perturbation is a rational number weakens the argument of [11].

[3] Note that we divide by π, not 2π as in the sine-Gordon model.

Moreover, one also checks that charges of different type $i \neq j$ commute. For instance, one has

$$\partial\varphi_1(z)e^{i\frac{4\pi}{\beta_2}\phi_2(z)}\partial\varphi_2(w)e^{-i\frac{4\pi}{\beta_1}\varphi_1(w)} = \left[\frac{i}{\beta_1(z-w)} + \partial\varphi_1\right]\left[\frac{i}{\beta_2(z-w)} + \partial\varphi_2\right]e^{i\frac{4\pi}{\beta_2}\varphi_2(z)}e^{-i\frac{4\pi}{\beta_1}\varphi_1(w)}$$

so the simple pole has residue

$$i\left[\left(1 - \frac{4\pi}{\beta_2^2}\right)\frac{\partial\varphi_2}{\beta_1} + \frac{\partial\varphi_1}{\beta_2}\right]e^{4i\pi[\frac{\varphi_2}{\beta_2} - \frac{\varphi_1}{\beta_1}]}$$

which is a total derivative on the manifold (19).

Again, while the commutation relations are established to first order in perturbation theory, simple scaling arguments show that they hold to any order. We conclude that, to any order in perturbation, the theory exhibits the symmetry $\widehat{sl_{q_1}}(2) \otimes \widehat{sl_{q_2}}(2)$.

Under the condition (19), it is lengthy but straightforward to show the existence of non trivial local conserved quantities (this has first been shown in [7]). Besides the stress energy tensor, the next non trivial one has dimension 4, and follows from $\bar{\partial}T_4 = \partial H_4$, where

$$T_4 = a_1^2(1 - 3a_1^2)(\partial\varphi_1)^4 - \left(\frac{1}{4} - \frac{9}{4}a_1^2 - 4a_1^4\right)a_1^2(\partial^2\varphi_1)^2 + (1 \to 2) + 6a_1^2a_2^2(\partial\varphi_1\partial\varphi_2)^2 \tag{24}$$

and $a_i^2 = \frac{\beta_i^2}{4\pi}$.

The existence of this non trivial conserved quantity is expected to be enough, following standard arguments [12] to guarantee quantum integrability. We can make an easy guess for the particle excitations and the factorized S matrix. Commutation with the two quantum affine algebras independently dictates a tensor product structure. Moreover, since T_1, T_2 must always have the same parity, excitations should be massive particles carrying a **pair** of quantum numbers T_1, T_2: ie they are simultaneously solitons of the field ϕ_1 and ϕ_2. One then obtains as the simplest choice

$$S = -S_{\widehat{\beta_1}} \otimes S_{\widehat{\beta_2}}, \tag{25}$$

where S_β is the S-matrix of solitons for the ordinary sine-Gordon model at coupling β,

$$\frac{\widehat{\beta_i}^2}{8\pi} = \frac{\beta_i^2}{\beta_i^2 + 2\pi}, \tag{26}$$

and we used the results of [8]. This scattering matrix was first found in [7] using a different argument. It is important to stress that we do not have independently solitons of the field ϕ_1 and of the field ϕ_2, but composite objects. Also, the coupling in the S-matrices is renormalized non trivially. This is largely the consequence of the $\partial\varphi$ term appearing in the conserved currents, a feature which is not there in the standard sine-Gordon model, but is necessary here because the interaction is the product of two cosines: although the S-matrix is factorized, the theory is thus highly interacting. It is interesting to discuss the action and the factorized S matrix (25) from the point of view of a fermion theory: see appendix C for more details.

Some special points can be identified on the integrable manifold, which can be solved independently. For instance the case $\beta_1 = \beta_2 = \sqrt{2\pi}$ can be recast, by forming linear combinations of the fields, into a problem of two decoupled sine-Gordon theories at their free fermion point, in agreement with (25). The case $\beta_1 = 0$ is also trivially solvable, and equivalent to one free boson and one sine-Gordon model at the free fermion point. This is a very different description from the one provided by (25), which involves in particular a sine-Gordon S matrix at the point $\widehat{\beta_2}^2 = \frac{16\pi}{3}$, ie the $N = 2$ supersymmetric point. Nevertheless, we will check that, when physical quantities are computed, they have identical expressions in both descriptions.

From the relation

$$\left[T_i, e^{i(\alpha\varphi_j + \bar{\alpha}\bar{\varphi}_j)}\right] = \delta_{ij}\frac{\beta_i}{2\pi}(\alpha - \bar{\alpha})e^{i(\alpha\varphi_j + \bar{\alpha}\bar{\varphi}_j)}, \tag{27}$$

we find the soliton operators (up to polynomials in derivatives of ϕ_1, ϕ_2)

$$\Psi_i^\pm = \exp\left[\pm i\frac{2\pi}{\beta_i}\varphi_i\right]$$
$$\bar{\Psi}_i^\pm = \exp\left[\mp i\frac{2\pi}{\beta_i}\bar{\varphi}_i\right], \tag{28}$$

which have topological charge $T_i = \pm 1$. The local current $J_i^\pm$ is a raising (lowering) operator for particles of type i: it annihilates solitons (antisolitons) and transforms antisolitons (solitons) into solitons (antisolitons). $J_i^\pm$ conserves $T_j, i \neq j$.

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B. Boundary integrability and new IR fixed points

The main step in [4] was to use the integrability of the boundary sine-Gordon model. Here, one can easily prove that the model (17) defined on the half line $x \in [-\infty, 0]$ with a boundary interaction at $x = 0$

$$\mathcal{B} = \lambda \int dy \cos \frac{\beta_1}{2} \phi_1(y) \cos \frac{\beta_2}{2} \phi_2(y), \tag{29}$$

is integrable for any choice λ, Λ. This model we call the double boundary sine-Gordon model (DBSG). In the factorized scattering picture, the boundary interaction is described by a reflection matrix solution of the boundary Yang Baxter equation. The factorized structure of the bulk S-matrix leads to the immediate conjecture for this reflection matrix

$$R = -R_{\widehat{\beta}_1} \otimes R_{\widehat{\beta}_2}, \tag{30}$$

where R is the reflection matrix of the boundary sine-Gordon model given explicitly in [5].

Of course the original Kane-Fisher impurity problem is massless in the bulk. As in the spinless case, we describe this problem starting from the massive theory with a bulk coupling (17) by using the quasiparticles basis, and letting $\Lambda \to 0$. In that limit, we get right and left moving massless solitons with dispersion relation $e = \pm p = m e^{\theta}$, with θ the rapidity. Movers of opposite types interact via a constant S-matrix, while movers of the same type still interact with (25). In addition, R movers bounce back as L movers at the boundary. We recall in particular the physical amplitudes

$$\left|(R_{\widehat{\beta}_i})^+_+\right|^2 = \frac{1}{1 + e^{-\frac{4\pi}{\beta_i^2}(\theta - \ln T_B)}}, \quad \left|(R_{\widehat{\beta}_i})^+_-\right|^2 = 1 - \left|(R_{\widehat{\beta}_i})^+_+\right|^2. \tag{31}$$

Here, T_B is an energy scale related with the coupling λ by $T_B \propto \lambda^2$, where we used (19).

Already at this point, the identification of these amplitudes allows a rough analysis of the infrared fixed point. If we expand the modulus square of the R-matrices in terms of the inverse coupling we find that $|R_{\widehat{\beta}_i}|^2$ expands in powers of $T_B^{-4\pi/\beta_i^2} \propto \lambda^{-8\pi/\beta_i^2}$. On the other hand, by the Fermi Golden rule, these are proportional to the square of the matrix elements of the operators perturbing the IR fixed point. We thus see that these operators must have dimension

$$d_i = \frac{2\pi}{\beta_i^2} + 1. \tag{32}$$

The same result is obtained when analyzing physical properties (see below).

This conclusion is, a priori, very surprising. Indeed, the standard analysis [3] of the IR limit based on the classical action (29) would suggest that, at large coupling, the fields ϕ_i are located at the bottom of the boundary potentials, leading to Dirichlet boundary conditions with

$$\phi_i(x = 0) = \frac{2\pi}{\beta_i} n_i, \ n_i + n_j = 0 \text{ mod } 2. \tag{33}$$

This fixed point we refer to in the following as the standard IR fixed point. With (33), it is easy to classify the allowed operators describing the approach to the standard IR fixed point. One simply computes the partition function of the model with these boundary conditions at $x = 0, x = L$, and matches with the formula of conformal invariant field theories, $Z = Tr w^{L_0 - c/24}$. Here, one finds

$$Z = \frac{1}{\eta(w)} {\sum_{n_1, n_2}}' w^{-2\pi \left(\frac{n_1^2}{\beta_1^2} + \frac{n_2^2}{\beta_2^2} \right)}, \tag{34}$$

where the primed sum indicates that n_1, n_2 have the same parity, $w = e^{-\pi/LT}$. The dimensions of the allowed operators, up to integers, are thus of the form

$$d = 2\pi \left(\frac{n_1^2}{\beta_1^2} + \frac{n_2^2}{\beta_2^2} \right).$$

In particular, the lowest dimension operators have dimension $\frac{8\pi}{\beta_i^2}, \frac{2\pi}{\beta_1^2} + \frac{2\pi}{\beta_2^2}$. Remarkably, the ones with $x = \frac{2\pi}{\beta_i^2} + 1$ are not there!

There are several explanations for this discrepancy. The first, of course, is the possibility that our S and R matrices have not been correctly identified. The previous arguments about affine quantum group symmetries are however usually considered to be quite strong. They can be made even stronger at some special points with $N = 2$ supersymmetry (see below). Moreover, physical results for all limiting cases where the solutions is known by other means will be reproduced correctly (see also below). Finally, the S and R matrices also agree with perturbation theory around the free fermion point $\beta_i^2 = 2\pi$ (see appendix C). If we do believe these S and R matrices, another possible explanation is that the perturbation around the IR fixed point has a vanishing radius of convergence, so the expansion of (31) (or other physical quantity) in powers of T_B is the result of a resummation, and cannot be used to naively identify the dimension of the perturbing operator. Such phenomenon has been observed in other integrable models [13].

We suspect however that this discrepancy arises, more simply, because the IR fixed point obtained by naive analysis of the action is not the correct one. To justify this, we recall first that integrable flows are usually found to be integrable considered either as perturbations of the UV or of the IR fixed point, and with the same set of conserved quantities. For instance, in the boundary sine-Gordon model perturbed by $\cos \beta\phi$, the local conserved quantities like T_4 commute not only with the perturbation, but also with its "dual" $\cos \frac{8\pi}{\beta}\phi$. As a result, the flow near the IR fixed point, which is approached along this dual field plus local integrals of motion T_{2n}, is integrable too. In the present case, we checked that our T_4 quantity does commute, besides $\cos \beta_1\phi_1 \cos \beta_2\phi_2$, with the currents $J_i^\pm$, but **not** with $\cos \frac{8\pi}{\beta_i}\phi_i$, nor with $\cos \frac{4\pi}{\beta_1}\phi_1 \cos \frac{4\pi}{\beta_2}\phi_2$. As a result, it seems impossible to flow to the disconnected fixed point, since its approach would not be integrable. On the other hand, the IR fixed point described above was found to be approached along operators of the same dimension as $J_i^\pm$. Very presumably, it is approached along these operators indeed (plus the T_{2n}), providing an integrable flow.

As another piece of justification, it is useful to consider a generalization of (17) given by the action

$$S = \frac{1}{2} \int dx dy \left[(\partial_\mu \phi_1)^2 + (\partial_\mu \phi_2)^2 + (\partial_\mu \phi)^2\right] + \frac{\Lambda}{2} \int dx dy \left[e^{\beta\phi} \cos(\beta_1\phi_1 + \beta_2\phi_2) + e^{-\beta\phi}\cos(\beta_1\phi_1 - \beta_2\phi_2)\right] \quad (35)$$

involving thus one more bosonic field. Non local conserved currents generalizing (18) can be built

$$J_i^\pm(z) = (i\beta_i \partial\varphi_i \pm \beta\partial\varphi) \, e^{\pm i\frac{4\pi}{\beta_j}\varphi_j} \quad (36)$$

provided the condition

$$\beta_1^2 + \beta_2^2 - \beta^2 = 4\pi, \quad (37)$$

holds. By the same arguments, the model is integrable. An integrable boundary action follows

$$\mathcal{B} = \frac{\lambda}{2} \int dy \left[e^{\beta\phi(0)/2} \cos\left(\frac{\beta_1\phi_1 + \beta_2\phi_2}{2}(y)\right) + e^{-\beta\phi(0)/2}\cos\left(\frac{\beta_1\phi_1 + \beta_2\phi_2}{2}(y)\right)\right]. \quad (38)$$

It is possible to compute the algebra generated by the conserved charges and it is found to be the same as in the previous model. Thus, the S and R matrices for this model are essentially the same as the ones for the previous, simpler model with $\beta = 0$, with however the the new condition (37). The IR fixed point deduced from the classical action is the "standard" one like for $\beta = 0$ since the field ϕ cannot be compactified.

This generalization is useful because at the value $\beta_1^2 = 4\pi$, the model (35) coincides with the bosonized form of the $N = 2$ supersymmetric sine-Gordon model, as can easily be inferred from the fact that the J_1 currents have dimension 3/2: $J_1^\pm \equiv G^\pm$. The bulk perturbation is unitary and preserves supersymmetry [14]. The same is true for the boundary perturbation. This supersymmetry is a particular case of the quantum affine symmetry generated by the currents $J_i^\pm$. The preservation of supersymmetry is a more firmly based property than the preservation of quantum affine symmetries however, because the generators are local. We now use the conservation of supersymmetry to justify the identification of the IR fixed point provided by the S and R matrices. Let us first recall what happens for the supersymmetric point of the ordinary boundary sine-Gordon model first [15]. Then, $\beta^2 = \frac{16\pi}{3}$. The perturbing operator $\cos \beta\phi$ in the UV has boundary dimension $x = 2/3 = \frac{1}{2} + \frac{1}{6}$ corresponding to the superpartner of the chiral primary field X, and preserves supersymmetry. The IR fixed point is, in that case, correctly reproduced by the standard analysis of the classical action. The Dirichlet boundary conditions analogous to (33) give rise to the partition function

$$Z = \frac{1}{\eta(w)} \sum_n w^{\frac{3n^2}{2}} = \chi_{00}^{NS}, \quad (39)$$

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which is the Neveu Schwartz character of the identity (that one is in the NS sector follows immediately from the fact that boundary conditions are the same on both ends of the cylinder). The IR fixed point is approached along the operator $\cos\frac{8\pi}{\beta}\phi$ of dimension $x = 3/2$, ie along the operators $G^\pm$. This is very natural from a Landau Ginzburg point of view: near the minima of the potential, the effective potential is $W = X^2$ with no left-over chiral primary field in the spectrum beside the identity. The approach has thus to be along a descendent of the identity, with $G^\pm$ as the natural candidates. It is not totally obvious a priori that a perturbation along $G^\pm$ preserves supersymmetry; while G^+ anticommutes with G^+, the anticommutator of G^+ and G^- is non zero. However, the $G^\pm$ are fermionic operators; to have a bosonic IR action, they must thus be coupled to some fermionic boundary degrees of freedom, and these allow the restoration of supersymmetry [15].

We can now get back to the model (38) when $\beta_1^2 = 4\pi$, ie with unbroken $N = 2$ supersymmetry. It is easy to check that the standard IR fixed point (33) cannot be approached while preserving supersymmetry. On the other hand, the leading operators describing the approach to our new IR fixed point have dimensions $d = 3/2$ and $d = 1 + \frac{2\pi}{\beta_2^2}$. The first one corresponds to $J_1^\pm = G^\pm$, and, like for the ordinary sine-Gordon model, it does preserve supersymmetry after addition of fermionic boundary degrees of freedom. The second one corresponds to $J_2^\pm$. Because the two affine quantum algebras in the problem commute, it too does preserve supersymmetry. Hence, our fixed point is perfectly compatible with unbroken supersymmetry, while the standard one is not.

Away from the supersymmetric points, a similar argument can be made based on the conservation of affine quantum symmetries instead [4]. It leads right away to the claim that the IR fixed point should be approached along the non local conserved currents: $J^\pm \propto e^{\pm 8i\pi\phi/\beta}$ in the ordinary sine-Gordon case, as is well known, and along $J_i^\pm$, of dimensions $d_i = 1 + \frac{2\pi}{\beta_i^2}$, in the double sine-Gordon case, in agreement with the conjectured S and R matrices .

Based on the analysis of physical properties (see below and appendix E), we conjecture that the operator content of the IR fixed point is a **subset** of the spectrum encoded in

$$Z = \frac{1}{\eta^2(w)} \sum_{n_1} w^{-2\pi\frac{n_1^2}{\beta_1^2}} \sum_{n_2} w^{-2\pi\frac{n_2^2}{\beta_2^2}} \qquad (40)$$

Note that the partition function (40) would correspond to a model with Dirichlet boundary conditions of the type (33) but with no coupling between n_1 and n_2. A possible conjecture for the ratio of boundary entropies in the UV and IR is

$$\frac{g_{IR}}{g_{UV}} = \left(\frac{\beta_1^2}{\beta_1^2 + 2\pi}\frac{\beta_2^2}{\beta_2^2 + 2\pi}\right)^{1/2} \qquad (41)$$

while the same ratio for the standard fixed point would read $\left(\frac{g_{IR}}{g_{UV}}\right)_{standard} = 2\left(\frac{\beta_1^2}{8\pi}\frac{\beta_2^2}{8\pi}\right)^{1/2}$. If this result is correct, our new fixed point is less stable than the open (separate leads) one: therefore, it might well be that it is reached only with the UV action (8) on the integrable manifold $g_\rho + g_\sigma = 2$.

The question of course remains: why is the identification of the IR fixed point misleading in our case, while it was not for the boundary sine-Gordon model? We have no satisfactory answer to that question at the present time. But there is an interesting indication: in sharp distinction with the ordinary sine-Gordon model, the double sine-Gordon model (17) seems to not be integrable classically, as discussed in the appendix A. Integrability in the present case is thus a truly quantum property. Since integrability is also responsible (via the analysis of non local conserved currents and the resulting S and R matrices) of the unexpected nature of the IR fixed point, it is maybe not so surprising that this IR fixed point cannot be obtained from a classical analysis of the action, and thus differs from the standard one.

C. Summary

We can now get back to the physical problem. A comparison of (16) with the results of this section leads to the identification of the integrable manifold in terms of the g-factors: $g_\rho + g_\sigma = 2$. The postulated exact S-matrix has the form of a tensor product of spin and charge degrees of freedom

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[4]This is a little tricky to do in practice, because of non locality problems for general values of β_i. It is however possible for special values of β_i [16].

$$S = -S_\rho \otimes S_\sigma, \tag{42}$$

with $S_{\rho,\sigma}$ the S-matrices of a sine-Gordon model with parameters $\frac{\beta^2}{8\pi} = \frac{g}{g+1}$. Similarly the boundary reflection matrix is given by

$$R = -R_\rho \otimes R_\sigma. \tag{43}$$

The elementary excitations of the folded theory are quasiparticles carrying a charge number $Q_\rho = \pm 1$ (in units where $e = 1$) and a spin number $Q_\sigma = \pm 1$ (in units where a physical electron has $S^z = \pm \hbar$). Here, we have set

$$Q_\rho = \frac{1}{\pi} \frac{1}{\sqrt{\pi g_\rho}} \int dx \ \partial_x \theta_\rho$$
$$Q_\sigma = \frac{1}{\pi} \frac{1}{\sqrt{\pi g_\sigma}} \int dx \ \partial_x \theta_\sigma \tag{44}$$

Having a complete description of the kinematics of the model, we can now compute physical properties. We will restrict to the $T = 0$ case here. Extensions to finite T are in principle possible, but quite technical.

IV. DC CONDUCTANCE AT $T = 0$

We will exactly follow the same line of argument as in [4]. The quasiparticles are considered as a gas interacting through factorized scattering. Standard thermodynamics arguments can be applied, in the presence of electric and magnetic chemical potentials, to compute the densities and the filling fractions of these quasiparticles. These are expressed via the thermodynamic Bethe ansatz. Having these densities, a rate equation can then be written to compute the contribution to the conductances of the current backscattered by the impurity.

A. General computation.

We only consider the case of an external voltage, so the quasiparticles have chemical potential $e^{\pm V/2T}$, T the temperature. At $T = 0$, a Fermi sea is formed, consisting of positively charged quasi-particles, with either spin up or down. These quasiparticles carry charge labels $(1, 1)$ and $(1, -1)$. Due to the factorized form of the S matrix, the scattering of these quasiparticles is factorized into the scattering of the first and second labels. The first labels scatter diagonally with the element $(S_\rho)^{++}_{++}$. The second labels in general scatter non diagonally. They would scatter diagonally if g_σ was of the form $1/integer$, but then bound states (carrying a non vanishing electrical charge) would have to be taken into account. It is in fact simpler to analyze the more general case corresponding to model (35), taking $g_\rho = p_1$, $g_\sigma = p_2$, p_i integers. The final results can then be analytically continued to the region of interest (19) with $\beta = 0$ (a computation right at $\beta = 0$ is also possible, although more complicated. We will describe this elsewhere). The scattering of the magnetic quantum numbers is then non diagonal. The TBA is slightly intricate to write: one needs to introduce pseudoparticles to diagonalize this scattering. The final result fortunately is very much like in the ordinary sine-Gordon model [17,18], and is conveniently encoded in the TBA diagram
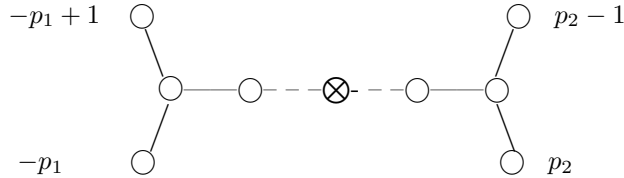
The TBA equations are

$$\epsilon_0 = \frac{1}{2} e^\theta + \frac{k}{2\pi} * \left[\epsilon_1^- + \epsilon_{-1}^- \right]$$
$$\epsilon_i = \frac{k}{2\pi} * \left[\epsilon_{i-1}^- + \epsilon_{i+1}^- \right]$$
$$\epsilon_{-p_1}^- = 0, \ \epsilon_{-p_1+1}^- = -p_1 \frac{V}{2} + \frac{k}{2\pi} * \epsilon_{-p_1+2}^- \tag{45}$$

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In these equations, the star indicates a convolution of the functions with the kernel $k(\theta) = \frac{1}{2\cosh\theta}$, and ϵ^- designates the negative part of ϵ [18]. It is a simple exercise in Fourier transform to extract from (45) an equation satisfied by ϵ_0 itself. Setting $\epsilon \equiv -\epsilon_0^-$, one has

$$\epsilon(\theta) - \int_{-\infty}^{A} \Phi(\theta - \theta')\epsilon(\theta')d\theta' = \frac{V}{2} - \frac{1}{2}e^\theta \tag{46}$$

Here, the kernel Φ has fourier transform

$$\tilde{\Phi} = \tilde{k}^2 \left(K_{-p_1+2} + K_{p_2-2} \right) \tag{47}$$

where $\tilde{k}(x) = \frac{1}{2\cosh x}$, the kernel K_{-p_1+2} arises from the left part of the diagram, and coincides with the derivative of the logarithm of $(S_\rho)_{++}^{++}$, and the kernel K_{p_2-2} arises from the right part of the diagram

$$K_{-p_1+2} = 2\cosh x \frac{\sinh(p_1-1)x}{\sinh p_1 x}, \quad K_{p_2-2} = 2\cosh x \frac{\cosh(p_2-2)x}{\cosh(p_2-1)x} \tag{48}$$

leading to

$$\tilde{\Phi} = 1 - \frac{\cosh(p_1 + p_2 - 1)x}{2\sinh p_1 x \cosh(p_2 - 1)x}. \tag{49}$$

We recall that ϵ describes the excitation energy of new quasiparticles which do not have a well defined magnetic quantum number - this is because the scattering of magnetic numbers is non diagonal. The total density of these quasiparticles is obeys

$$\rho(\theta) - \int_{-\infty}^{A} \rho(\theta')\Phi(\theta - \theta')d\theta' = \frac{1}{2h}e^\theta, \tag{50}$$

where we have reinstated Planck's constant. This density follows simply from the excitation energy as $\rho = -\frac{1}{h}\frac{d\epsilon}{d\theta}$.

In the previous equations, A is the Fermi rapidity defined by $\epsilon(A) = 0$; it is the edge of the Fermi sea. A can be determined, and the foregoing equations solved, by using a standard Wiener Hopf analysis. If we write $K(\theta) = \delta(\theta) - \Phi(\theta)$, we have

$$\tilde{K}(\omega) = \frac{1}{N(\omega)N(-\omega)}, \tag{51}$$

with

$$N(\omega) = \sqrt{\frac{2\pi}{g_\rho}} \frac{\Gamma(i\omega/2)e^{i\omega\Lambda}}{\Gamma(i\omega g_\rho/2)\Gamma(1/2 + i\omega(g_\sigma - 1)/2)}, \tag{52}$$

a function analytic in the lower half plane when we choose $\Lambda = 1/2\log 2 + (g_\sigma - 1)/2\log(g_\sigma - 1)/2 + g_\rho/2\log g_\rho/2$. Here the conventions for the Fourier transforms are

$$g(\theta) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega\theta} \tilde{g}(\omega), \quad \tilde{g}(\omega) = \int_{-\infty}^{\infty} d\theta e^{i\omega\theta} g(\theta). \tag{53}$$

Then following exactly the same steps as in [4] we find the solution for the fourier transform of the density

$$\tilde{\rho}(\omega) = \frac{1}{2h} \frac{e^{(1+i\omega)A}}{(1 + i\omega)} N(-i)N(\omega). \tag{54}$$

The excitation energy of the particles follows

$$\tilde{\epsilon}(\omega)e^{-i\omega A} = -i\frac{qV}{2} \frac{N(0)N(\omega)}{\omega} + \frac{ie^A}{2} \frac{N(-i)N(\omega)}{\omega - i}. \tag{55}$$

The condition $\epsilon(A) = 0$ is equivalent to

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$$\lim_{\omega \to \infty} \omega \tilde{\epsilon}(i\omega) e^{-i\omega A} = 0, \tag{56}$$

which in turns gives the explicit value of the Fermi rapidity, A

$$e^A = eV \frac{N(0)}{N(-i)}. \tag{57}$$

In the presence of the impurity, the backscattered current is computed by using a Landauer Büttiker approach [19]. The only difference with the computation done in [4] is that the quasiparticles have a magnetic degree of freedom. However, we are only interested here in the properties of the electric charge, so we simply sum over all possible channels of reflection for the magnetic quantum number. This leads to the final expression for the current

$$I(V, T_B) = e \int_{-\infty}^{A} d\theta \rho(\theta) \, |\, (R_\rho)_+^+ (\theta - \theta_B)|^2. \tag{58}$$

This expression was derived under the assumption g_ρ, g_σ integers. We expect however that it holds for any values of these constants, in particular for the values on the manifold $g_\rho + g_\sigma = 2$. Indeed, physical quantities in integrable models have (so far) always had a smooth behaviour in terms of the coupling constants, despite the very different possible Bethe ansätze that depend on their arithmetic properties. Furthermore, Fateev [7] has computed some thermodynamic properties for the more general model (35). One can check on his computations, by comparison with perturbative results or sigma models approximations, that the results are indeed analytical in terms of the coupling constants.

Let us now analyze (58). In the UV limit, the matrices R_+^+ become unity and we find the result

$$I(V, T_B = 0) = \frac{e^2 V}{h} g_\rho, \tag{59}$$

as expected [5]. The full result at finite T_B is given by a double expansion, one for each reflection matrix. The modulus squared of the previous reflection matrices is given by

$$|\,(R_\rho)_+^+ (\theta)|^2 = \frac{1}{1 + \exp(-\frac{2}{g_\rho})\theta}, \quad |\,(R_\sigma)_+^+ (\theta)|^2 = \frac{1}{1 + \exp(-\frac{2}{g_\sigma})\theta} \tag{60}$$

For large voltage, we just expand the previous expression and evaluate the current at each order by suming over residues, we obtain the following strong coupling expansion

$$I(V, T_B) = \frac{e^2 V}{2h} \sum_{n=1}^{\infty} I_{2n} \left(\frac{eV}{T_B} \right)^{2n/g_\rho} \tag{61}$$

with

$$I_{2n} = (-1)^{n+1} N(0) \frac{N(-2in/g_\rho)}{(1 + 2n/g_\rho)} \left(\frac{N(0)}{N(-i)} \right)^{2n/g_\rho}. \tag{62}$$

Using a similar procedure and treating the case in which $T_B < V$, we can also expand the reflection matrices by splitting the integral over the rapidity, θ in two pieces. After a little algebra, we find the weak coupling expansion

$$I(V, T_B) = \frac{e^2 V}{2h} N(0) PP \int \frac{d\omega}{2\pi} \frac{N(\omega)}{(1 + i\omega)} \left\{ \sum_{n=0}^{\infty} (-1)^{n+1} \frac{b^n}{2n/g_\rho + i\omega} - \frac{i\pi g_\rho}{2} \frac{b^{-i\omega g_\rho/2}}{\sinh(\pi \omega g_\rho/2)} \right\} \tag{63}$$

with $b = (T_B e^{-A})^{2/g_\rho}$. We integrate the first sum by closing the contour in the lower half plane and the second term by closing in the upper half plane. In the second term, the zeroes of $N(\omega)$ cancels the poles of $\sinh(\pi \omega g_\rho/2)$ and only the poles $\omega = 2ni$ with n integer, and $\omega = i$ contributes. We get

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[5]To get the correct version, it suffice to rescale V as mentionned at the beginning

$$I(V, T_B) = \frac{e^2 V}{h} g_\rho + \sum_{n=1}^{\infty} U_{2n} \left(\frac{T_B}{eV} \right)^{2n} + U_1 T_B, \tag{64}$$

with the coefficients

$$U_{2n} = \frac{e^2 V}{2h} \sqrt{\frac{2\pi}{g_\rho}} \pi g_\rho \frac{(-1)^n}{n!} e^{-2n\Lambda} \frac{N(0)(\frac{N(-i)}{N(0)})^{2n}}{\Gamma[-ng_\rho]\Gamma[1/2 - n(g_\sigma - 1)](2n-1)\sin(\pi n g_\rho)}$$

$$U_1 = -\frac{e\pi g_\rho}{2h} \sin(\frac{\pi}{2} g_\rho). \tag{65}$$

Let us write the first few orders more explicitly

$$I(V, T_B) = \frac{e^2 V}{h} g_\rho \left\{ 1 - \frac{\pi}{2} \sin(\frac{\pi g_\rho}{2}) \frac{T_B}{eV} + \frac{\pi^{5/2}}{g_\rho^2} \frac{\Gamma(g_\rho)}{\Gamma(1/2 - g_\rho)} \frac{1}{\Gamma(g_\rho/2)^2 \Gamma(g_\sigma/2)^2} \left(\frac{T_B}{eV} \right)^2 + \cdots \right\} \tag{66}$$

The spin conductance in a magnetic field follows from the previous analysis simply by the interchange of g_σ and g_ρ. In order to compare with perturbation theory and complete the picture, we need to establish a relation between the boundary scale T_B and the perturbative coupling λ. This is done in appendix D and we find

$$T_B = \frac{\pi}{2\sin(\frac{\pi}{2} g_\rho)} \lambda^2. \tag{67}$$

B. Special Cases.

When $g_\sigma = g_\rho = 1$ the sum (61) can be done analytically, in that case we have that

$$I_{2n} = \frac{(-1)^{n+1}}{2n+1} \tag{68}$$

and we find the result

$$I(V, T_B) = \frac{e^2 V}{h} - \frac{eT_B}{h} \arctan \frac{eV}{T_B}. \tag{69}$$

This result can be found by other means since by writing the action, not in the charge and spin fields basis but rather in the spin up and down basis, the action decouples into two Luttinger actions which do not interact with each other. Moreover each is at the free fermion, or Toulouse, point and the resulting conductance is well known to be of the form given by (69). The two expressions can be precisely matched using (67) together with the results of [4][6].

Another special point is $g_\rho = 2$ and $g_\sigma = 0$. The spin field then totally decouples, while the dynamics of the charge field is described by the ordinary boundary sine-Gordon model at the free fermion point. One thus expects a conductance of the form (69) (with however an overall factor of 2, and a renormalized T_B). On the other hand, the S matrix which we claim describes the double sine-Gordon model is not at the free fermion point in the charge sector, but rather at the $N = 2$ supersymmetric point! Nevertheless, special cancellations then occur to reproduce the correct physical quantities. Indeed, using the expression (62), we have at that particular point

$$I_{2n} = (-1)^{n+1} \frac{N(-in)}{1+n} \left(\frac{2}{N(-i)} \right)^n. \tag{70}$$

In the limit $g_\rho \to 2$, this vanishes for n odd. For n even, one has

$$I_{4n} \approx 2 \frac{(-)^{n+1}}{2n+1} \left(\frac{1}{\sin \pi g_\rho/2} \right)^{2n} \tag{71}$$

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[6]Most unfortunately, there are two misprints in the latter reference. In eq. (6.20), the left hand side should have a $2\lambda A_1$ instead of λ_1. The next equation should have a term $eV/2T_B$ instead of eV/T_B in brackets.

This reproduces the correct result using (67).

A last special point is $g_\rho = 0, g_\sigma = 2$. There, one expects the spin field to essentially decouple as the fluctuations of the charge field become very large. Explicit computation using (64) gives rise to

$$\frac{I}{g_\rho} \approx \frac{e^2 V}{h}\left(1 + \sum_{n=1}^{\infty}(-)^n \frac{\sqrt{\pi}}{n!\Gamma(\frac{1}{2}-n)(\frac{1}{2}-n)}\left(\frac{\pi T_B}{2eV}\right)^{2n}\right) \tag{72}$$

This coincides indeed with the classical limit of the spinless problem [4].

V. CONCLUSION

The main result of our study is the existence of a new kind of IR fixed point in this impurity problem. This fixed point has a vanishing transmitted current but it does not correspond to the open (separate leads) fixed point as demonstrated by the analysis of the operator content. Of course, since the transmitted current vanishes, it does not correspond to any of the other fixed points (eg current transmitting and spin reflecting, or partially transmitting and partially reflecting) considered so far [1,3]. The main property of our fixed point is that it is approached along operators which describe the transfer of one electron charge but no spin, or one spin 1/2 but no charge. This is clearly seen by reformulating the non local currents $J_i^{\pm}$ in terms of the physical fields; a quick identification can be made using the dimensions $d = 1 + \frac{1}{g_\rho}$ and $d = 1 + \frac{1}{g_\sigma}$ (recall that for the separate leads fixed point, the approaching operators have $d = \frac{1}{g_\rho} + \frac{1}{g_\sigma}$, corresponding to the transfer of a physical electron carrying a spin 1/2, and $d = \frac{4}{g_\rho}$, corresponding to the transfer of a pair of electrons with opposite spin). Understanding the physical meaning of these properties in more details is an open challenge. A possibility is that particles with half the electron charge and half its spin are actually transfered: "half electrons" have actually been encountered in the double barrier case, where their appearance corresponds to the transfer through an "island". To make further progress, a reliable formula for the boundary entropy would be most useful - technical difficulties have to be resolved before this can be obtained, however.

It is not clear to us how the existence of these new fixed points changes the conclusions of [1,3]. Our analysis, however, indicates that an instanton approach to the study of the problem might be misleading. In particular, the duality property, which is the main ingredient of the lore in the spinless case (although only partially understood yet), cannot always hold in the case with spin: the UV and IR expansions of our current do not match under the transformations $g \to 4/g$.

APPENDIX A: NON-INTEGRABILITY IN THE CLASSICAL CASE.

In this appendix we want to show that the system described by the action

$$S = \int \sum_{i=1}^{2} \frac{1}{2}(\partial_\mu \phi_i)^2 + \Lambda \cos(\beta_1 \phi_1)\cos(\beta_2 \phi_2) \tag{A1}$$

does not have a spin 4 conserved quantity classically, and is presumably non integrable.

Using the light-cone variables $z = x + iy$, $\bar{z} = x - iy$, the classical equations of motion are given by

$$\partial_z \partial_{\bar{z}} \phi_1 = -\frac{\Lambda \beta_1}{4}\sin(\beta_1 \phi_1)\cos(\beta_2 \phi_2),$$
$$\partial_z \partial_{\bar{z}} \phi_2 = -\frac{\Lambda \beta_2}{4}\cos(\beta_1 \phi_1)\sin(\beta_2 \phi_2). \tag{A2}$$

The first non trivial condition of integrabilty is the existence of a spin 4 conserved current, T_4 such that

$$\partial_{\bar{z}} T_4 = \partial_z \theta_2. \tag{A3}$$

The most general form of the T_4 current respecting the $U(1)$ symmetries is given by

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$$T_4 = A_1(\partial_z\phi_1)^4 + A_2(\partial_z\phi_2)^4 + A_3(\partial_z\phi_1)^2(\partial_z\phi_2)^2 + A_4(\partial_z^2\phi_1)^2 + A_5(\partial_z^2\phi_2)^2. \tag{A4}$$

When we take the $\partial_{\bar{z}}$ derivative of T_4, several of the terms can be eliminated up to total ∂_z derivatives, so it is enough to look for a θ_2 of the form

$$\theta_2 = B_1(\partial_z\phi_1)^2 \cos(\beta_1\phi_1)\cos(\beta_2\phi_2) + B_2(\partial_z\phi_2)^2 \cos(\beta_1\phi_1)\cos(\beta_2\phi_2) +$$
$$B_3(\partial_z\phi_1)(\partial_z\phi_2)\sin(\beta_1\phi_1)\sin(\beta_2\phi_2). \tag{A5}$$

Straightforward algebra gives the relations

$$-A_1\Lambda\beta_1 = -\beta_1 B_1, \quad -A_2\Lambda\beta_2 = -\beta_2 B_2$$
$$-A_3\frac{\Lambda\beta_1}{2} = -\beta_1 B_2 + \beta_2 B_3, \quad -A_3\frac{\Lambda\beta_2}{2} = -\beta_2 B_1 + \beta_1 B_3 \tag{A6}$$
$$-A_4\frac{\Lambda\beta_1^2}{2} = 2B_1, \quad A_4\frac{\Lambda\beta_1\beta_2}{2} = B_3$$
$$A_5\frac{\Lambda\beta_1\beta_2}{2} = B_3, \quad -A_5\frac{\Lambda\beta_2^2}{2} = 2B_2.$$

If $\beta_1 \neq \beta_2$, then these equations have no solutions and thus the system has no non trivial conserved quantity of spin 4 classically. Presumably, it then does not have conserved quantities of higher spin either.

The algebra in the quantum case is a bit different. The problem is to find a chiral quantity T_4, of the most general form

$$T_4 = A_1(\partial_z\varphi_1)^4 + A_2(\partial_z\varphi_2)^4 + A_3(\partial_z\varphi_1)^2(\partial_z\varphi_2)^2 + A_4(\partial_z^2\varphi_1)^2 + A_5(\partial_z^2\varphi_2)^2. \tag{A7}$$

such that the residue of the simple pole in the short distance expansion with the perturbation is a total derivative. Using the propagators $<\varphi_i(z)\varphi_j(z')> = \frac{-\delta_{ij}}{4\pi}\ln(z-z')$, one finds that the quantity T_4 as given in the text (24) solves the problem.

That some $1+1$ quantum field theories can be non integrable classically, but are quantum mechanically integrable at finite values of Planck's constant (β_i here) was observed before: see for instance [20]. Usually however, quantum integrability becomes "classical" once the theory is fermionized. While a similar phenomenon occurs here for the supersymmetric case [21], we are not aware of a classically integrable, partially fermionized, version of our model for $\beta_1^2 + \beta_2^2 = 4\pi$.

APPENDIX B: BOUNDARY QUANTUM INTEGRABILITY

We show that the boundary perturbation in the action (16) preserves quantum integrability. It is sufficient to prove that the boundary perturbation of its conformal limit preserves the bulk integrable manifolds. The proof follows Ghoshal and A.Z. Zamolodchikov argument in [5]. The unperturbed conformal theory is constituted by the two independent bosons ϕ_1, ϕ_2 in the half plane, with Neumann boundary conditions

$$\frac{\partial}{\partial x}\phi_i(0,y) = 0 \tag{B1}$$

such that the propagators turn out to be

$$\langle\phi_i(x_1,y_1)\phi_j(x_2,y_2)\rangle = -\delta_{ij}\left[\frac{1}{4\pi}\log\frac{(y_1-y_2)^2+(x_1-x_2)^2}{\kappa^2} + \frac{1}{4\pi}\log\frac{(y_1-y_2)^2+(x_1+x_2)^2}{\kappa^2}\right], \tag{B2}$$

where κ is a short distance cutoff. The half-line bosons $\phi_i(x,y)$ can thus be considered as the folding of two full line bosons $\Phi_i(x,y)$,

$$\phi_i(x,y) = \frac{1}{\sqrt{2}}\left[\Phi_i(x,y) + \Phi_i(-x,y)\right]. \tag{B3}$$

The full line left and right components $\Phi_i(x,y) = \Phi_{i,L}(x,y) + \Phi_{i,R}(x,y)$ allow the further decomposition of the half-line bosons in a left and a right component

$$\phi_i(x,y) = \Phi_i^e(x,y) + \Phi_i^e(-x,y) \tag{B4}$$

where we define the even fields

$$\Phi_i^e(x,y) = \frac{1}{\sqrt{2}} \left[\Phi_{i,L}(x,y) + \Phi_{i,R}(-x,y) \right] \tag{B5}$$

whose correlations are

$$\langle \Phi_i^e(x_1,y_1) \Phi_j^e(x_2,y_2) \rangle = -\delta_{ij} \frac{1}{4\pi} \log \frac{y_1 - y_2 + i(x_1 - x_2)}{\kappa}. \tag{B6}$$

Conformal invariance of the half-line model is guaranteed by the condition that no energy-momentum flux escapes the system across the boundary, $T(x,y) = \overline{T}(-x,y)$, where $T(x,y) = -2\pi \sum_i (\partial_z \phi_i)^2 = -2\pi \sum_i (\partial_z \Phi_i^e(x,y))^2$, $\overline{T}(x,y) = -2\pi \sum_i (\partial_{\bar{z}} \phi_i)^2 = -2\pi \sum_i (\partial_{\bar{z}} \Phi_i^e(-x,y))^2$, and $z = y + ix$.

This condition implies an infinite number of higher spin conservation laws, $T_{s+1}(x,y) = \overline{T}_{s+1}(-x,y)$, where the spin label s is an odd integer. Therefore a boundary perturbation preserves integrability if some of the conserved currents satisfy

$$T_{s+1}(x,y) - \overline{T}_{s+1}(-x,y) \big|_{x=0} = \frac{\partial}{\partial x} t_s(x). \tag{B7}$$

The standard argument [5] consists in evaluating the left-hand side at first order in conformal perturbation theory

$$\lambda \int dy' \left[T_{s+1}(x,y) - \bar{T}_{s+1}(-x,y) \right] \Phi(0,y'), \tag{B8}$$

where $\Phi(0,y) = \cos \beta_1/2\phi_1(y) \cos \beta_2/2\phi_2(y)$ is the perturbing operator. Therefore the OPE of the currents with the perturbing operator are needed. Since the half-line bosons at the boundary satisfy

$$\phi_i(0,y) = 2\Phi_i(0,y) \tag{B9}$$

it turns out that the computation of (B8) is identical to the one in the bulk for the quantity $\partial_{\bar{z}} T_{s+1}$ and the perturbing operator $\cos \beta_1 \phi_1(0,y) \cos \beta_2 \phi_2(0,y)$. As a result, the bulk integrable manifold gives rise to a boundary integrable one.

APPENDIX C: PERTURBATIVE COMPUTATION OF THE S MATRIX

In this appendix we give more credibility to the S matrix proposed in the text by checking that it is correctly recovered in perturbation. As shown in [7], the bosonic action of (17) can be put in a useful fermionic form by first introducing two auxiliary boson fields $\chi_{1,2} = \frac{1}{2\sqrt{\pi}}(\beta_1 \phi_1 \pm \beta_2 \phi_2)$ whose action

$$S = \int \left\{ \left(\frac{\pi}{2\beta_1^2} + \frac{\pi}{2\beta_2^2} \right) [(\partial \chi_1)^2 + (\partial \chi_2)^2] + \left(\frac{\pi}{2\beta_1^2} - \frac{\pi}{2\beta_1^2} \right) (\partial \chi_1)(\partial \chi_2) + \Lambda \left(\cos 2\sqrt{\pi} \chi_1 + \cos 2\sqrt{\pi} \chi_2 \right) \right\} \tag{C1}$$

is the bosonization of the two fermions theory defined by

$$S = \int \left\{ \sum_{i=1,2} \bar{\psi}_i (i\gamma_\mu \partial^\mu - m) \psi_i + g \sum_{i=1,2} (\bar{\psi}_i \gamma_\mu \psi_i)^2 + g_1 (\bar{\psi}_1 \gamma_\mu \psi_1)(\bar{\psi}_2 \gamma^\mu \psi_2) \right\}. \tag{C2}$$

The couplings are defined by $g = \frac{\pi}{2}(\frac{\pi}{\beta_1^2} + \frac{\pi}{\beta_2^2} - 1)$ and $g_1 = \pi(\frac{\pi}{\beta_1^2} - \frac{\pi}{\beta_2^2})$. The integrable manifold (19) becomes

$$g_1^2 = 2g(2g + \pi). \tag{C3}$$

The perturbative computation of the scattering amplitudes of the fermions permits their identification with the composite charge and spin solitons. Denoting by $(\rho \, \sigma)$ with $\rho, \sigma = \pm$ the four elementary excitations whose exact scattering matrix is (25), the identification is

$$\psi_1 = (++), \quad \bar{\psi}_1 = (--), \quad \psi_2 = (+-), \quad \bar{\psi}_2 = (-+). \tag{C4}$$

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The perturbative region $g \sim 0$ for the two fermions theory implies $\beta_i^2 \sim 2\pi$ and therefore each sine-Gordon component in the exact solution (25)

$$S^{(\rho'_1\sigma'_1)(\rho'_2\sigma'_2)}_{(\rho_1\sigma_1)(\rho_2\sigma_2)}(\theta) = -S^{\rho'_1\rho'_2}_{\rho_1\rho_2}(\theta;\hat{\beta}_1)S^{\sigma'_1\sigma'_2}_{\sigma_1\sigma_2}(\theta;\hat{\beta}_2) \tag{C5}$$

can be expanded [22] around the respective free fermion points $\hat{\beta}_i^2 \sim 4\pi$. We recall that the sine-Gordon couplings $\hat{\beta}_i$ are related to the couplings β_i of our model by $\hat{\beta}_i^2 = \frac{8\pi\beta_i^2}{\beta_i^2 + 2\pi}$. As an example let us consider the following expansion

$$S^{(--)(++)}_{(++)(--)}(\theta) = \pi^2(\frac{2\pi}{\beta_1^2} - 1)(\frac{2\pi}{\beta_2^2} - 1)\frac{1}{\sinh^2\theta} + o(2) = -\frac{2\pi g}{\sinh^2\theta} + o(2) \tag{C6}$$

where we have also used the integrability condition (C3). This result has to be compared with the direct perturbative evaluation of the amplitude $S^{\bar{\psi}_1\psi_2}_{\psi_1,\bar{\psi}_1}$. We will proceed with a dispersion relation method, as described in [22] for the Thirring model. The two perturbative couplings g and g_1 are treated as independent. The integrability condition will be imposed at the end of the second order computation which is needed because $o(g_1^2)$ still contribute to the lowest order in g once (C3) is taken into account. The tree level amplitudes, schematically in figure 1, are determined by the two four fermion vertices in the lagrangian contracted on the respective external fermions.
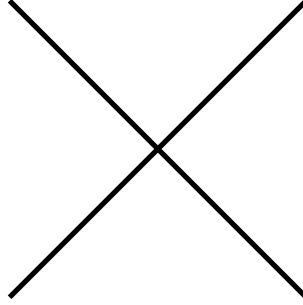

FIG. 1.   Tree level diagram.

Some of them are

$$S^{\psi_1\psi_1}_{\psi_1\psi_1}(\theta) = -(1 + i2g\frac{1 - \cosh\theta}{\sinh\theta}) \qquad\qquad S^{\psi_1\bar{\psi}_1}_{\psi_1\bar{\psi}_1}(\theta) = -(1 + i2g\frac{1 + \cosh\theta}{\sinh\theta}) \tag{C7}$$

$$S^{\bar{\psi}_1\psi_1}_{\psi_1\bar{\psi}_1}(\theta) = i4g\frac{1}{\sinh\theta}$$

$$S^{\psi_2\bar{\psi}_2}_{\psi_1\bar{\psi}_1}(\theta) = -ig_1\frac{1}{\sinh\theta} \qquad\qquad S^{\psi_2\psi_1}_{\psi_1\psi_2}(\theta) = -ig_1\frac{1}{\sinh\theta}$$

$$S^{\bar{\psi}_2\psi_2}_{\psi_1\bar{\psi}_1}(\theta) = ig_1\frac{1}{\sinh\theta} \qquad\qquad S^{\bar{\psi}_2\psi_1}_{\psi_1\bar{\psi}_2}(\theta) = ig_1\frac{1}{\sinh\theta}$$

$$S^{\psi_1\psi_2}_{\psi_1\psi_2}(\theta) = -(1 - ig_1\frac{\cosh\theta}{\sinh\theta}) \qquad\qquad S^{\psi_1\bar{\psi}_2}_{\psi_1\bar{\psi}_2}(\theta) = -(1 + ig_1\frac{\cosh\theta}{\sinh\theta}).$$

We have written on the same lines the crossing symmetric pairs. The other amplitudes can be obtained by the listed above exploiting either the symmetry $1 \leftrightarrow 2$ or CPT symmetry.

The real part of the second order contributions, schematically in figure 2, can be obtained by the cutting technique.
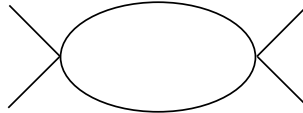

FIG. 2.   Second order diagram.

Actually it is not necessary to use explicitly the Cutkowski rule. One can use unitarity, as we show in the following example. Let's consider the amplitude $S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}$. According to unitarity of the total S-matrix $S^\dagger S = 1$ and elasticity of $2 \to 2$ particles scattering the real part of the second order contribution is determined by the tree level amplitudes

$$\mathrm{Re}(S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}) = \frac{1}{2}\left[(S^{\psi_1\bar\psi_1}_{\psi_1\bar\psi_1}+1)(S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1})^* + S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(S^{\bar\psi_1\psi_1}_{\bar\psi_1\psi_1}+1)^* + S^{\psi_2\bar\psi_2}_{\psi_1\bar\psi_1}(S^{\bar\psi_1\psi_1}_{\psi_2\bar\psi_2})^* + S^{\bar\psi_2\psi_2}_{\psi_1\bar\psi_1}(S^{\bar\psi_1\psi_1}_{\bar\psi_2\psi_2})^*\right] + o(3) \tag{C8}$$

$$= -8g^2\frac{1+\cosh\theta}{\sinh^2\theta} - g_1^2\frac{1}{\sinh^2\theta} + o(3).$$

In total generality a second order diagram in the class of figure 2 can be parameterized as

$$g^2\,\frac{i}{\sinh\theta}\,\frac{i\pi-\hat\theta}{\pi\sinh\hat\theta}\,[a\cosh^2\theta + b\cosh\theta + c] \tag{C9}$$

where $\hat\theta = \theta$, $i\pi-\theta$, $i\pi$ when the total momentum P which flows into the diagram is $P = (p_1+p_2)$, $\pm(p_1-p_2)$, 0, resp. Obviously the order g^2 can be substituted by g_1^2 and gg_1 depending on the kind of vertices involved in the diagram. The factor $1/\sinh\theta$ is the usual jacobian of the total momentum conservation expressed in the rapidities θ_1,θ_2 and the mandelstam variable is $s = (p_1+p_2)^2 = 2m^2(1+\cosh\theta)$. The function $r(\hat\theta) = \frac{i\pi-\hat\theta}{\pi\sinh\hat\theta}$ is the result of the dispersion integral on the two-particles phase space $\Theta(s-4m^2)/\sqrt{s(s-4m^2)}$

$$r(\hat\theta) = \frac{1}{\pi}\int_{4m^2}^\infty \frac{ds'}{s'-P^2}\frac{2m^2}{\sqrt{s'(s'-4m^2)}}. \tag{C10}$$

The polynomial in $\cosh\theta$ is the effect of the contraction of external fermion wave functions on the numerators of the loop fermion propagators. Each diagram can at most be logarithmic divergent at high energy, $\log s \sim \theta$. This fact constraints the polynomial to be of order not greater than two. Therefore the amplitude of our example can be written in the form

$$S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(\theta) = \frac{i4g}{\sinh\theta} + \tag{C11}$$

$$\frac{i8g^2}{\sinh\theta}\left\{r(\theta)[A\cosh^2\theta + B\cosh\theta + C] + r(i\pi-\theta)[\bar A\cosh^2\theta + \bar B\cosh\theta + \bar C] + r(i\pi)[D\cosh\theta + E]\right\} +$$

$$\frac{ig_1^2}{\sinh\theta}\left\{r(\theta)[A_1\cosh^2\theta + B_1\cosh\theta + C_1] + r(i\pi-\theta)[\bar A_1\cosh^2\theta + \bar B_1\cosh\theta + \bar C_1] + r(i\pi)[D_1\cosh\theta + E_1]\right\}$$

$$+o(3).$$

Some of the coefficients of the polynomials, which encode the freedom in the finite part of a regularized diagram, are determined by imposing i) the matching with the already computed real part (C9), ii) the asymptotic behaviour $S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(\theta) \to 0$ as $\theta \to \infty$ and iii) the crossing symmetry relation $S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(\theta) = S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(i\pi-\theta)$. The result is

$$S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(\theta) = \frac{i4g}{\sinh\theta} - \frac{8g^2}{\sinh^2\theta}(1+\cosh\theta) - \frac{g_1^2}{\sinh^2\theta} - \frac{i8g^2}{\pi\sinh\theta}\left(2\frac{\theta}{\sinh\theta}\cosh\theta - E\right) + \frac{ig_1^2}{\pi\sinh\theta}E_1 + o(3). \tag{C12}$$

The same procedure can be used for the amplitudes $S^{\psi_1\psi_1}_{\psi_1\psi_1}$ and $S^{\psi_1\bar\psi_1}_{\psi_1\bar\psi_1}$ which are mapped one into the other by crossing symmetry and whose asymptotic behaviour at high energy is $S(\theta) \to$ constant. We obtain

$$S^{\psi_1\psi_1}_{\psi_1\psi_1}(\theta) = -1 - \frac{i2g}{\sinh\theta}(1-\cosh\theta) + \frac{2g^2}{\sinh^2\theta}(1-\cosh\theta)^2 \tag{C13}$$

$$-\frac{i2g^2}{\pi\sinh\theta}\left(4\frac{\theta}{\sinh\theta} + d\cosh\theta + e\right) - \frac{ig_1^2}{\pi\sinh\theta}\left(\frac{\theta}{\sinh\theta} + d_1\cosh\theta + e_1\right) + o(3)$$

$$S^{\psi_1\bar\psi_1}_{\psi_1\bar\psi_1}(\theta) = -1 - \frac{i2g}{\sinh\theta}(1+\cosh\theta) + \frac{2g^2}{\sinh^2\theta}[(1+\cosh\theta)^2+4] + \frac{g_1^2}{\sinh^2\theta}$$

$$-\frac{i2g^2}{\pi\sinh\theta}\left(-4\frac{\theta}{\sinh\theta} - d\cosh\theta + e\right) - \frac{ig_1^2}{\pi\sinh\theta}\left(-\frac{\theta}{\sinh\theta} - d_1\cosh\theta + e_1\right) + o(3).$$

Imposing now the threshold behaviour $\lim_{\theta\to 0}S^{\psi_1\psi_1}_{\psi_1\psi_1}(\theta) = -1$ and $\lim_{\theta\to 0}(S^{\psi_1\bar\psi_1}_{\psi_1\bar\psi_1}(\theta) + S^{\bar\psi_1\psi_1}_{\psi_1\bar\psi_1}(\theta)) = 0$ we obtain the four relations

$$E = -\frac{d}{2}, \quad e = -4 - d, \quad E_1 = -2 - 2d_1, \quad e_1 = -1 - d_1. \tag{C14}$$

We are thus left with only two undetermined constants, d and d_1. This freedom reflects the freedom we have in the definition of the coupling constants g and g_1. Let us choose to define them through the asymptotic behaviour of the amplitudes

$$\lim_{\theta \to +\infty} S_{\psi_1 \psi_1}^{\psi_1 \psi_1}(\theta) = -e^{-i2g} \tag{C15}$$

$$\lim_{\theta \to +\infty} S_{\psi_1 \psi_2}^{\psi_1 \psi_2}(\theta) = -e^{-ig_1}$$

fixing the coefficients to $d = 0$, $d_1 = 0$.

The amplitude we want to compare with the expansion of the exact results (25) is therefore

$$S_{\psi_1 \bar{\psi}_1}^{\bar{\psi}_1 \psi_1}(\theta) = \frac{i4g}{\sinh \theta} - \frac{8g^2}{\sinh^2 \theta}(1 + \cosh \theta) - \frac{g_1^2}{\sinh^2 \theta} - \frac{i16g^2}{\pi \sinh \theta} \frac{\theta}{\sinh \theta} \cosh \theta - \frac{i2g_1^2}{\pi \sinh \theta} + o(3). \tag{C16}$$

On the integrable manifold (C3) at lowest order in g we obtain

$$S_{\psi_1 \bar{\psi}_1}^{\bar{\psi}_1 \psi_1}(\theta) = -\frac{2\pi g}{\sinh^2 \theta} + o(2) \tag{C17}$$

which coincides with the lowest order expansion of the exact result (C6). The same check can now be extended to the remaining amplitudes.

APPENDIX D: KELDYSH COMPUTATION.

In order to compute the differential conductance we need to use the Keldysh formalism since the system is driven by reservoirs. To do so, we will use the formulation of the model on the full line as described by the equations (12). The effect of the reservoirs can be implemented by shifting the charge fields, $\Theta_\rho \to \Theta_\rho + \sqrt{g_\rho/\pi} a(t)/2$ with $\partial_t a(t) = V$. Under this prescription the current is evaluated by taking the functionnal derivative

$$I(t) = -i\frac{\delta \log Z[a(t)]}{\delta a(t)}, \tag{D1}$$

where we have used conventions in which $\hbar = e = 1$. Using the Keldysh contour, $\mathcal{C}$, which goes from $-\infty$ to ∞ and then comes back (see figure 3), to expand the partition function we obtain to first order
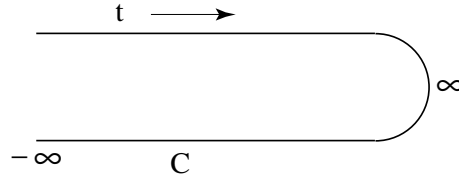

FIG. 3. Contour for the perturbative evaluation.

$$I^{(2)}(0) = -i\frac{\lambda^2 g_\rho}{8} \int_{\mathcal{C}} dt \, \sin[\frac{g_\rho}{2} a(t)] \left\{ P_{g_\rho + g_\sigma}^{\mu +}(t) + P_{g_\rho + g_\sigma}^{+\mu}(-t) \right\} \tag{D2}$$

where $\mu = \pm$ depending on the location of t, ie upper part of the contour or lower part of the contour. The functions $P_g^{\mu,\mu'}(t)$ are the corresponding contraction of the vertex operators time ordered on the Keldysh contour

$$P_g^{\pm\pm}(t) = \frac{1}{(\delta \pm |t|)^{g/2}}, \quad P_g^{\pm\mp}(t) = \frac{1}{(\delta \mp t)^{g/2}}. \tag{D3}$$

To this order the result can be found explicitely, it is given by

$$I^{(2)}(0) = -\frac{\pi}{2\Gamma[(g_\rho + g_\sigma)/2]}\left(\frac{g_\rho}{2}\right)^{(g_\rho+g_\sigma)/2}|\lambda|^2 V^{\frac{g_\rho+g_\sigma}{2}-1}. \tag{D4}$$

This is in agreement with the Bethe ansatz solution given in the bulk of the text provided we make the identification (putting $h = 2\pi$ and $e = 1$ in the TBA expressions and $g_\rho + g_\sigma = 2$ in the previous perturbative expression)

$$|\lambda|^2 = \frac{2}{\pi}\sin(\frac{\pi}{2}g_\rho)T_B. \tag{D5}$$

APPENDIX E: OPERATOR CONTENT OF THE IR FIXED POINT.

To get a better knowledge of the operator content of the IR fixed point, it is useful to study correlation functions of various operators. This is done using form-factors [23]. The form factors for theories with a factorized S matrix of the form (25) were analysed by Smirnov [24] for the massive theory described in the bulk of the text. These are now matrix elements of operators $\mathcal{O}$ in the space of states created by the Fadeev-Zamolodchikov operators

$$\langle 0|\mathcal{O}(0,0)Z_{\epsilon_1}^*(\theta_1)\cdots Z_{\epsilon_n}^*(\theta_n)|0\rangle \tag{E1}$$

where now the asymptotic states are described by isotopic indices giving the two quantum numbers of the excitation: $\epsilon_i = (T_1, T_2)$. Here we will concentrate on the current form factors. For these operators, the massless limit, which is needed to treat the boundary perturbation problem, has basically the same form has for the boundary sine-Gordon case discussed in [23]. Here, the theory has $U(1) \otimes U(1)$ symmetry, and we can construct currents invariant with respect to either of these $U(1)$. There is an infinity of non-zero matrix elements, or form factors but for the sake of the argument we need only the two solitons ones. Let us concentrate on the right moving currents. For the field ϕ_1 one has

$$f_+^1(\theta, \theta')_{\epsilon,\epsilon'} = d_{\beta_1,\beta_2}e^{(\theta+\theta')/2}\frac{\zeta_{\beta_1,\beta_2}(\theta-\theta')}{\cosh\frac{1}{2\beta_1}(\theta'-\theta-i\pi)\sinh\frac{1}{2\beta_2}(\theta'-\theta-i\pi)}T_1 \tag{E2}$$

and for the field ϕ_2,

$$f_+^2(\theta, \theta')_{\epsilon,\epsilon'} = d_{\beta_1,\beta_2}e^{(\theta+\theta')/2}\frac{\zeta_{\beta_1,\beta_2}(\theta-\theta')}{\sinh\frac{1}{2\beta_1}(\theta'-\theta-i\pi)\cosh\frac{1}{\beta_2}(\theta'-\theta-i\pi)}T_2. \tag{E3}$$

The constant d_{β_1,β_2} and the function ζ_{β_1,β_2} are given in [24] and will not be needed in the following. The computation of current-current correlator in the Luttinger model follows immediatly using the methods of [23]. By using the explicit expression written before, it is easy to check that $< 0|\partial_\pm\phi_1\partial_\pm\phi_2|0 >= 0$ as expected. The zero temperature, frequency dependent, conductance would then follow from a correlator of the form $< 0|\partial_+\phi_1\partial_-\phi_1|0 >$, after the usual identification $\theta_\rho \to \phi_1$. Using the boundary state formulation, the previous form factors lead to a contribution to that correlator which is of the form

$$\int \frac{d\theta d\theta'}{(2\pi)^2 2!}|f_+^1(\theta, \theta')|^2 e^{(z-\bar{z})(e^\theta + e^{\theta'})} \tag{E4}$$

$$[P_{\hat{\beta}_1}(\theta_{1B})P_{\hat{\beta}_1}(\theta_{2B}) - Q_{\hat{\beta}_1}(\theta_{1B})Q_{\hat{\beta}_1}(\theta_{2B})][P_{\hat{\beta}_2}(\theta_{1B})P_{\hat{\beta}_2}(\theta_{2B}) + Q_{\hat{\beta}_2}(\theta_{1B})Q_{\hat{\beta}_2}(\theta_{2B})],$$

where we used the notation $\theta_{iB} = \theta_i - \theta_B$ and $\theta_B = \log T_B$. $P_{\hat{\beta}}$ and $Q_{\hat{\beta}}$ are the sine-Gordon elements of the reflection matrix R are coupling g. We have $P = R_+^+ = R_-^-$ and $Q = R_-^+ = R_+^-$. The frequency dependence follow from a Fourier transformation of the previous expression over time. We then get one single integral and the frenquency dependence can be recast in the reflection matrices by shifting the rapidities as was done in [23]. The expansion around the IR fixed point then follows from the expansion of the reflection matrices. Doing this we obtain a multiple power series with the following exponents: $(\omega/T_B)^{2n}$, $(\omega/T_B)^{4\pi n/\beta_1^2}$ and $(\omega/T_B)^{4\pi n/\beta_2^2}$. Assuming the current correlator explores the whole operator content of the theory, this is consistent with (40)

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